248. Proposed by E. T. BELL, Seattle, Washington.

If $u_{n+2} = 4u_{n+1} - u_n$, with $u_0 = 2$, $u_1 = 4$, prove that the $\triangle_n$, whose sides are $u_n - 1$, u_n, $u_n + 1$, has an integral area; also that all triangles, $\triangle_n$, whose areas are integers, and whose sides are consecutive integers, are given by this process. Hence show that, as n increases, the area $\triangle_n$ approximates $(\sqrt{3}/4)u_n^2$, and find the degree of approximation.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Integrating,

$$u_n = C_1(2 + \sqrt{3})^n + C_2(2 - \sqrt{3})^n. \tag{1}$$

When $n = 0$, $u_0 = 2$, and when $n = 1$, $u_1 = 4$; then

$$2 = C_1 + C_2 \cdots \tag{2}$$

$$4 = C_1(2 + \sqrt{3}) + C_2(2 - \sqrt{3}). \tag{3}$$

(2) and (3) give $C_1 = 1$, $C_2 = 1$; so (1) becomes

$$u_n = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n \tag{4}$$

Taking u_n as the second of three consecutive numbers, $u_n - 1$, $u_n + 1$ are the others, and the area

$$\begin{aligned}\triangle_n &= \tfrac{1}{4}\sqrt{3\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\}^2[\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\}^2 - 2^2]}\\ &= \tfrac{1}{4}\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\}\sqrt{3[\{(7+4\sqrt{3})^n + (7-4\sqrt{3})^n\} - 2]}\cdots.\end{aligned} \tag{5}$$

The expression under the radical sign must be shown to be a perfect square. For $n = 1, 2$, etc., this is the case. For the extreme case,

$$\lim_{n\doteq\infty} \triangle_n = \lim_{n\doteq\infty} \tfrac{1}{4}\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\}^2\sqrt{3}\sqrt{1 - \frac{4}{(2+\sqrt{3})^n + (2-\sqrt{3})^n}}$$

$$= \tfrac{1}{4}\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\}^2\sqrt{3} = \tfrac{1}{4}\sqrt{3}u_n^2.$$

249. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A perfect number is a number which is equal to the sum of all its different divisors. In an old book on mathematics, the following method is given without proof for determining perfect numbers. The number $2^{n-1}(2^n - 1)$ is a perfect number, if $2^n - 1$ is a prime number. Prove the formula.

SOLUTION BY MRS. ELIZABETH BROWN DAVIS, U. S. Naval Observatory, Washington, D. C.

The sum of all the divisors of 2^{n-1}, including unity, is

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-3} + 2^{n-2} = 2^{n-1} - 1.$$

If $2^n - 1$ is prime, the sum of all the divisors of $2^{n-1}(2^n - 1)$ is equal to the sum of all the divisors of 2^{n-1}, including unity, plus the product of the sum of these divisors into $(2^n - 1)$, plus 2^{n-1}. Hence, the sum of all the divisors of

$$\begin{aligned}2^{n-1}(2^n - 1) &= 2^{n-1} - 1 + (2^{n-1} - 1)(2^n - 1) + 2^{n-1}\\ &= 2.2^{n-1} - 1 + (2^{n-1} - 1)(2^n - 1)\\ &= 2^n - 1 + (2^{n-1} - 1)(2^n - 1)\\ &= 2^{n-1}(2^n - 1).\end{aligned}$$

Therefore, $2^{n-1}(2^n - 1)$ is a perfect number. Q. E. D.

Also solved by H. N. CARLETON, ELIJAH SWIFT, J. H. WEAVER, J. W. BALDWIN, E. B. ESCOTT, H. C. FEEMSTER, and J. W. CLAWSON.